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Some Aspects of the Dynamics and Stability of Spinning Flexible Spacecraft

Engineering Science Operations
The Aerospace Corporation
El Segundo, Calif. 90245

12 September 1977

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THE AEROSPACE CORPORATION

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The hybrid coordinate approach combines discrete coordinates describing the translations and rotations of some bodies or reference frames of the system with distributed or modal coordinates describing the small relative motions of other parts of the system. This approach is illustrated by application to a simplified spinning spacecraft system. The example illustrates many features of flexural systems engaged in spin. The origins of specific terms in the dynamical equations and their anticipated effects on dynamics are discussed. → next page

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Literal stability criteria for the wobble motions of a simplified spacecraft system are developed. These criteria can be used to duplicate results of previously published studies involving considerably more sophisticated models as a point of departure. The stability criteria can be of value in preliminary design but are not intended to supplant more detailed followup studies to verify ultimate system stability.

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I. INTRODUCTION

This report is one of a series that deals with the problem of spinning flexural dynamic systems. In this initial document, a general overall description of the hybrid coordinate approach is provided as an introduction only. The general structure and explicit development of the equations have been provided in the literature.[1-3] A hybrid coordinate approach is illustrated by applying it to a simplified spacecraft system so that the general developments appearing in the literature can be better appreciated by potential users. A simple example is selected to illustrate many features of flexural systems engaged in spin. The origins of specific terms in the dynamical equations and their anticipated effects upon the dynamics are discussed.

A stability analysis is provided for a simplified spacecraft system. The emphasis is on the development of literal stability criteria for the wobble motions of the simplified spacecraft system that can be of value in preliminary design phases. Admittedly, some risk is associated with extrapolation of results for simplified dynamical systems to more complicated examples. The stability criteria can be used to duplicate results of previously published studies involving considerably more sophisticated models as a point of departure, however. The stability criteria can be of value in preliminary design phases but are not intended to supplant more detailed followup studies to verify ultimate system stability.

---

[1] P. W. Likins, Dynamics and Control of Flexible Space Vehicles, Jet Propulsion Laboratory, Technical Report No. 32-1329, Revision 1, January 1970.

[2] P. W. Likins, Finite Element Appendage Equations for Hybrid Coordinate Dynamic Analysis, Jet Propulsion Laboratory, Technical Report No. 32-1525, October 1971.

[3] P. W. Likins and P. H. Wirsching, "Use of Synthetic Modes in Hybrid Coordinate Dynamics Analysis," AIAA J., 6, 10: 1867-1872, October 1968.

A. HYBRID COORDINATE APPROACH

Traditional modal analyses entailing modal deformation coordinates of the entire vehicle have been employed for certain classes of spacecraft with much success. These approaches possess the advantage of being rigorous as long as small deformations are observed and can be computationally efficient in that irrelevant high frequency response can be truncated from the simulation. Traditional modal analyses entailing modal deformation coordinates of the entire vehicle may not be applicable to a larger class of space systems frequently encountered: systems of time-varying configuration, systems employing nonlinear control systems, systems employing discrete dampers, and systems requiring accommodation of large arbitrary vehicle motions.

Faced with these obstacles, some analysts have resorted to discrete parameter formulations. The total system, including flexural appendages, is idealized as a collection of rigid subbodies articulated to each other in some topological tree configuration. Equations of motion are then derived for this system. In this approach, such obstacles as large configuration variations, inclusion of discrete dampers, nonlinear controls, or large arbitrary vehicle motions can be accommodated. The physical system analyzed may not be particularly representative of the actual system, however. Satisfactory simulation may require many rigid bodies. The analyst in this case is faced with a complex task of formulating equations of motion for many rigid bodies and is forced to carry along irrelevant high frequency response in any simulation attempts.

The hybrid coordinate approach represents a natural compromise between fully discrete and exclusively modal methods. The most general definition of the hybrid coordinate approach to formulating equations of motion is

> any approach that combines discrete coordinates describing the translations and rotations of some bodies or reference frames of the system with distributed or modal coordinates describing the small relative motions of other parts of the system . . . .

The hybrid coordinate approach is applicable when all or part of the vehicle admits the assumption of small linearly elastic deformations. The most efficient simulation combines discrete coordinates with modal coordinates, retaining the generality of discrete coordinates where necessary and securing the computational advantages of modal coordinates where possible.

The literature devoted to the hybrid coordinate approach received a major impetus in 1970 with the publication of a JPL report entitled, Dynamics and Control of Flexible Space Vehicles, by P. W. Likins.[1] At that stage of development, the method presented was formally restricted to systems in which each flexible appendage is attached only to a rigid body or to several rigid bodies interconnected so that relative motion cannot induce deformation of the flexible appendage. The method as presented is best suited to systems in which the flexural appendage is the outer member of a system of interconnected bodies of topological tree configuration. Any number of flexural subsystems could be accommodated subject to the restrictions, and furthermore the appendage could be engaged in spin.

The dynamics of these general spacecraft configurations were derived using an Eulerian formulation. Generalized matrix expressions were developed and presented. These formidable general expressions can be specialized to particular cases as long as they fall within the stated restrictions. Since the publication of Ref. 1, several contributions have developed the idea of a hybrid coordinate approach even further.[2-4]

B. SCOPE OF WORK

The following effort parallels to some extent the work of Likins and Barbera.[5] It is hoped that this report might serve as a first step toward a

[4] P. W. Likins and G. E. Fleischer, Large-Deformation Modal Coordinates for Nonrigid Vehicle Dynamics, Jet Propulsion Laboratory, Technical Report No. 32-1565, November 1972.

[5] P. W. Likins and F. J. Barbera, Attitude Stability of Spinning Flexible Spacecraft, UCLA Engineering Report No. 7176, December 1971; also NASA-CR-123569.

comprehensive understanding of the dynamics of spinning flexural systems. In this document, essential features of flexural systems engaged in spin are discussed. Since we aim to develop a general awareness of the more important features of flexural dynamical systems, the discussion is directed toward the nonexpert in this area. Hopefully, this report and subsequent documents will serve as a palatable transition to the literature in the field.

II. DYNAMICAL FORMULATION

In this section, a particularly simple example is analyzed. This example, despite its simplicity, reveals many features of flexural systems engaged in spin. The origins of specific terms in the dynamical equations and their anticipated effects upon the dynamics are discussed.

The example consists of a rigid hub and a pair of mass particle/massless elastic spring combinations mounted symmetrically on the hub. The mass particle/massless elastic spring combinations are assumed to be representative of the flexural subsystem. The nature of the interconnection between the flexural subsystem and the rigid portion is optional. A variety of different interconnections are assumed: radially mounted subsystems, anticantilever subsystems, and an intermediate combination of the two extremes. We will show that the nature of the interconnection can be important and can influence the dynamics in a predictable way. It is further assumed that this entire system is engaged in spin. The nominal spin state is pure spin about a symmetry axis of the vehicle. This system is illustrated in Figure 1.

The equations of motion for the appendage are derived in Section II. A. The effects of spin are discussed in Section II. B. In Section II. C, the equations of motion for the entire system are derived, based on the radial mounting of the flexural subsystem to the rigid hub. Again, important properties of the overall system are discussed.

A. DYNAMICAL FORMULATION OF APPENDAGE SUBSYSTEM

The simple model, shown in Figure 2, consists of a rigid core to which two particles are attached through springs. The principal axes of the system in its nominal state remain coincident with the principal axes of the core, as the particles, each of mass m, are symmetrically located a distance $|\underline{r}_i|$ along the axis $\hat{y}$. The particles are allowed to move radially

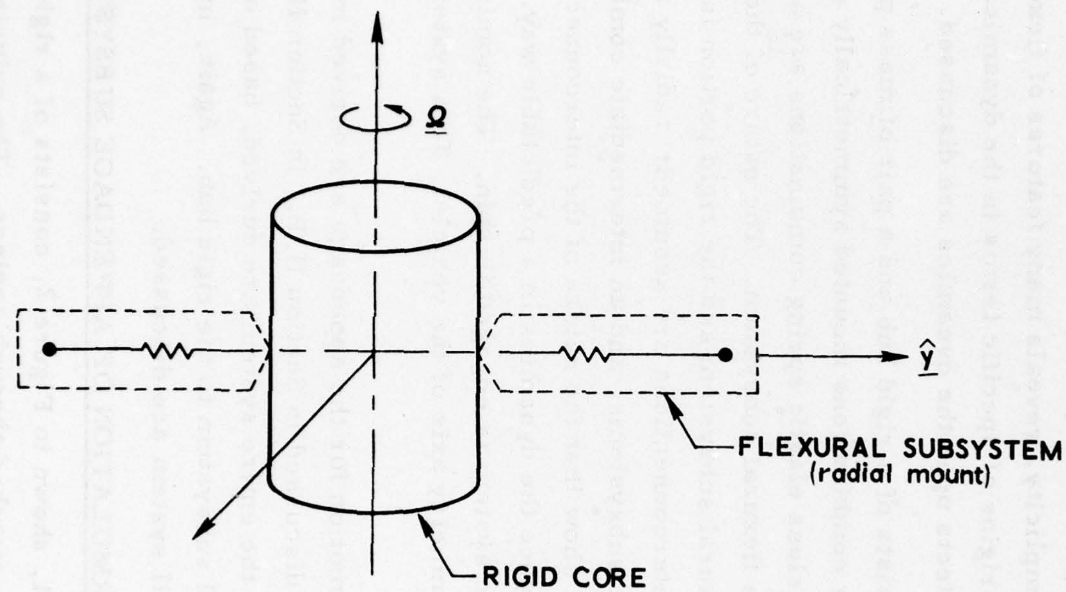

Figure 1. Simplified Spacecraft Configuration

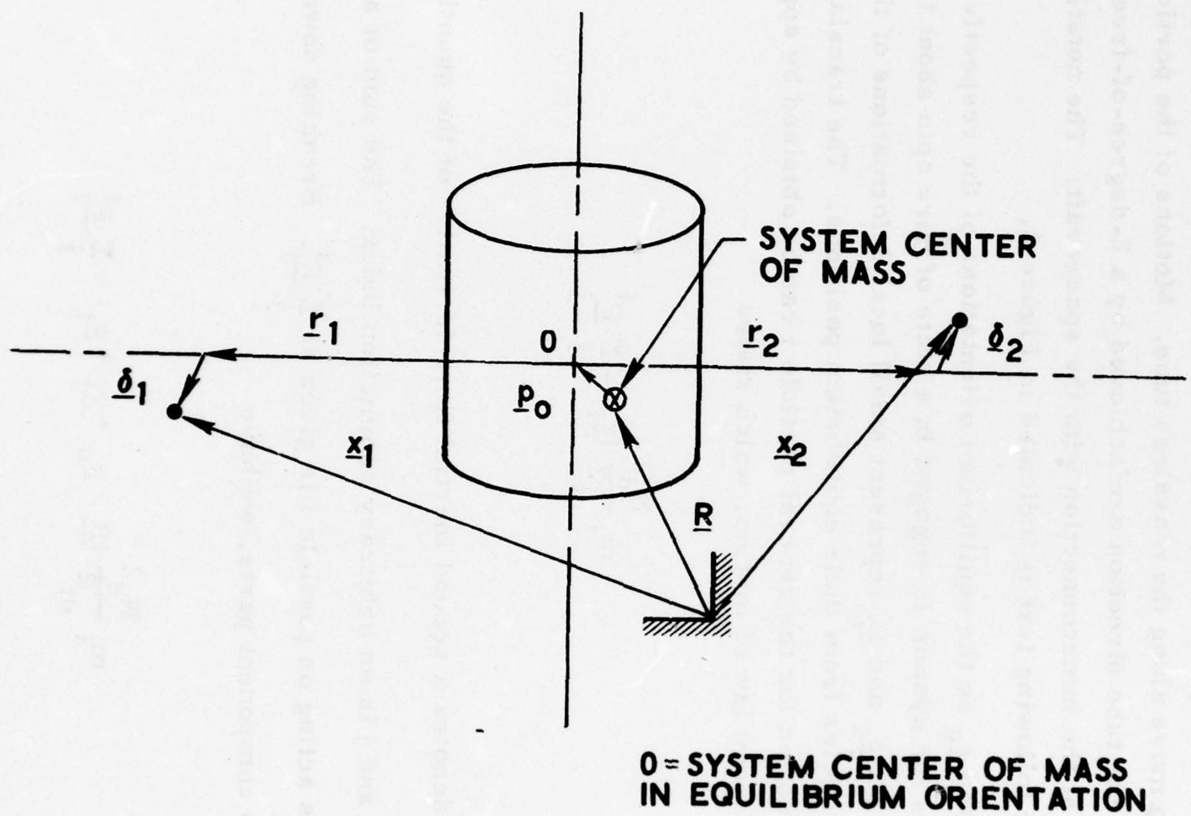

Figure 2. Notational Definition

along the $\hat{y}$ axis, as well as rotationally through a 2-degree-of-freedom pivot at the attachment point, thus allowing deformations in the other two axes as well. Conceptually, we might visualize the particle as being constrained to move along the massless tube. Motions of the particle transverse to the tube direction are achieved by a 2-degree-of-freedom hinge located at the interconnection with the spacecraft. The notation to be used in the following text is indicated in Figure 2.

Let $\underline{r}_1$ and $\underline{r}_2$ be the equilibrium orientations of the respective mass particles when the system is engaged in a state of pure spin about the symmetry axis. Let $\underline{\delta}_1$ and $\underline{\delta}_2$ represent small local deformations of the respective particles from their equilibrium positions. The translational equations of motion for the general particle i can be obtained by application of Newton's second law of motion, which states

$$m_i \frac{{}^N d^2}{dt^2} (\underline{x}_i) = \sum_j \underline{F}_j^i \tag{1}$$

where $\frac{{}^N d^2}{dt^2}(\ )$ denotes a second inertial time derivative of the quantity in parentheses, and j is an arbitrary summation index. The sum of all contact forces acting on particle i is given as $\sum_j \underline{F}_j^i$. Breaking down the vector $\underline{x}_i$ into component parts, we have

$$m_i \frac{{}^N d^2}{dt^2} (\underline{R} + \underline{\rho}_o + \underline{r}_i + \underline{\delta}_i) = \sum_j \underline{F}_j^i \tag{2}$$

By definition of the mass center, we have

$$m_T \underline{\rho}_o + \sum_i m_i (\underline{r}_i + \underline{\delta}_i) = \underline{0} \tag{3}$$

By symmetry and equality of masses, we have

$$\Sigma m_i \underline{r}_i = 0$$

Thus, the expression for $\underline{\rho}_o$ reduces to

$$\underline{\rho}_o = - \frac{\Sigma\, m_i\, \underline{\delta}_i}{m_T} \tag{4}$$

Further assuming that we are dealing with a force-free system yields

$$\frac{{}^N d^2}{dt^2} \underline{R} = \underline{0} \tag{5}$$

As a result of these assumptions, Eq. (2) can be expressed as

$$m_i \frac{{}^N d^2}{dt^2} \left(\underline{\delta}_i - \sum_{j=1}^{2} \frac{m_j}{m_T} \underline{\delta}_j + \underline{r}_i \right) = \sum_j \underline{F}_j^i \tag{6}$$

where $\sum_j \underline{F}_j^i$ represents the vector sum of all contact forces acting on particle i. Again, it is assumed that there are no net external forces acting upon the total system.

Define an orthonormal triad of unit vectors $(\hat{x}, \hat{y}, \hat{z})^T$ that are fixed in the rigid hub. The inertial angular velocity of this basis is designated by $\underline{\omega}$. The vectors $\underline{r}_i$ are invariant with respect to this basis, but the vectors $\underline{\delta}_i$ are not invariant in this basis. Carrying out the indicated differentiation by successive application of the Coriolis theorem,* we obtain

$$\frac{{}^N d}{dt}(\ ) = \frac{{}^P d}{dt}(\ ) + \underline{\omega} \times (\ ) \tag{7}$$

and letting $m = m_1 = m_2$, we find that Eq. (6) reduces to

$$\begin{aligned} m \Bigg\{ & \overset{\circ\circ}{\underline{\delta}}_i + 2\underline{\omega} \times \overset{\circ}{\underline{\delta}}_i + \dot{\underline{\omega}} \times \underline{\delta}_i + \underline{\omega} \times (\underline{\omega} \times \underline{\delta}_i) \\ & - \sum_{j=1}^{2} \frac{m}{m_T} \left[\overset{\circ\circ}{\underline{\delta}}_j + 2\,\underline{\omega} \times \overset{\circ}{\underline{\delta}}_j + \dot{\underline{\omega}} \times \underline{\delta}_j + \underline{\omega} \times (\underline{\omega} \times \underline{\delta}_j) \right] \\ & + \dot{\underline{\omega}} \times \underline{r}_i + \underline{\omega} \times (\underline{\omega} \times \underline{r}_i) \Bigg\} = \sum_j \underline{F}_j^i \end{aligned} \tag{8}$$

where ($^{\circ}$) denotes differentiation with respect to a rotating frame of reference and is a shorthand notation for the operator $\frac{{}^P d}{dt}(\ )$. In the following, we are interested in developing perturbation expressions about a nominal case of pure spin. By application of the angular velocity chain rule, we can write

$$\underline{\omega} = \underline{\Omega} + \underline{\delta\omega} \tag{9}$$

where $\underline{\Omega}$ is the nominal angular velocity of the system in its equilibrium state of pure constant spin. The terms retained in the following expressions are consistent with first approximation analysis. Products of small quantities,

---

* The operator $\frac{{}^P d}{dt}(\ )$ designates differentiation of the quantity in parentheses with respect to a rotating vector basis P. The inertial angular velocity of that basis is $\underline{\omega}$.

such as deformations and variational angular velocities, are considered second-order small.

The first-order approximation to Eq. (8) is

$$
\begin{aligned}
m\Big[\overset{\circ\circ}{\underline{\delta}}_i - \sum_j \frac{m}{m_T}\overset{\circ\circ}{\underline{\delta}}_j + 2\underline{\Omega}\times\overset{\circ}{\underline{\delta}}_i - 2\sum_j \frac{m}{m_T}\underline{\Omega}\times\overset{\circ}{\underline{\delta}}_j \\
+ \underline{\Omega}\times(\underline{\Omega}\times\underline{\delta}_i) - \sum_j \frac{m_j}{m_T}\underline{\Omega}\times(\underline{\Omega}\times\underline{\delta}_j)\Big] - \underline{F}_e^i \\
= -m\Big[\dot{\underline{\delta\omega}}\times\underline{r}_i + \underline{\Omega}\times(\underline{\Omega}\times\underline{r}_i) + \underline{\delta\omega}\times(\underline{\Omega}\times\underline{r}_i) \\
+ \underline{\Omega}\times(\underline{\delta\omega}\times\underline{r}_i)\Big] + \underline{F}'^i \qquad \underline{i} = 1,\ 2
\end{aligned}
\tag{10}
$$

where the readily identifiable homogeneous terms in deformations have been grouped, and summations over j are over j = 1 and 2.

A portion of the contact forces attributable to the elastic restoring forces has been partitioned and grouped on the left-hand side. This contribution is considered proportional to the deformations $\underline{\delta}_i$ from the equilibrium configuration, and the "weighting" matrix that maps these deformations into an elastic force $\underline{F}_e$ is referred to as the stiffness matrix. The remaining contribution to the contact forces is designated $\underline{F}'^i$ and is referred to as preload effects.

The equation formulation is encumbered by the presence of terms that reflect the effects of mass center shifts, i.e., $\sum \frac{m}{m_T}(\ )$. By a simple transformation of variables, those terms involving the mass ratio can be removed in a rigorous treatment. In fact, such an approach has much value because it emphasizes some important features about flexural systems. Consider the transformation

$$\underline{\xi} = \underline{\delta}_1 - \underline{\delta}_2$$

$$\underline{\eta} = \underline{\delta}_1 + \underline{\delta}_2 \tag{11}$$

Applying this transformation pair to Eq. (10) yields Eqs. (12) and (13). By differencing the two vector Eqs. (10) and making the identification in Eq. (11), we have

$$\begin{aligned} m\left[\ddot{\underline{\xi}} + 2\underline{\Omega} \times \dot{\underline{\xi}} + \underline{\Omega} \times (\underline{\Omega} \times \underline{\xi})\right] - \underline{F}_e^1 + \underline{F}_e^2 \\ = -2m\,[\underline{\delta\dot{\omega}} \times \underline{r}_1 + \underline{\Omega} \times (\underline{\Omega} \times \underline{r}_1) + \underline{\delta\omega} \times (\underline{\Omega} \times \underline{r}_1) \\ + \underline{\Omega} \times (\underline{\delta\omega} \times \underline{r}_1)] + \underline{F'}^1 - \underline{F'}^2 \end{aligned} \tag{12}$$

and by adding the two vector Eqs. (10) and making the identification in Eq. (11), we have

$$\begin{aligned} m\left[\ddot{\underline{\eta}} - 2\frac{m}{m_T}\ddot{\underline{\eta}} + 2\underline{\Omega} \times \dot{\underline{\eta}} - 4\frac{m}{m_T}\underline{\Omega} \times \dot{\underline{\eta}} \right. \\ \left. + \underline{\Omega} \times (\underline{\Omega} \times \underline{\eta}) - 2\frac{m}{m_T}\underline{\Omega} \times (\underline{\Omega} \times \underline{\eta})\right] - \underline{F}_e^1 - \underline{F}_e^2 \\ = \underline{F'}^1 + \underline{F'}^2 \end{aligned} \tag{13}$$

Two observations are made. First, the terms attributed to mass center shifts are absent in Eq. (12). Second, only Eq. (12)

contains forcing terms that reflect general spacecraft motion.* If we further identify the correspondence of the variable $\underline{\xi}$ to an asymmetric mode and the variable $\underline{\eta}$ to a symmetric mode, then the observations take on new meaning. Symmetric mode shapes are not likely to be excited by rotational motions of the hub. Conversely, an excited symmetric mode does not usually produce rotational motions of the hub. The absence of coupling with spacecraft rotational motions is consistent in this example. Another appealing fact is that terms attributable to mass center shifts appear only in Eq. (13). Mass center shifts do not occur with asymmetric mode shapes. Section II. B. discusses the effects that arise primarily as a consequence of spin. Equation (12) in particular is dealt with at some length.

B. EFFECTS OF SPIN

The left-hand portion of Eq. (12) is repeated for convenience:

$$m \overset{\circ\circ}{\underline{\xi}} + \underbrace{m\, 2\underline{\Omega} \times \overset{\circ}{\underline{\xi}}}_{\text{Coriolis}} + \underbrace{m\, \underline{\Omega} \times (\underline{\Omega} \times \underline{\xi})}_{\text{Centripetal}} - \underline{F}_e^1 + \underline{F}_e^2 \tag{14}$$

Two terms in particular arise as a consequence of the dynamics formulation: Coriolis and centripetal acceleration effects. To better appreciate the role of these effects, we coordinatize the equations in a basis fixed in the hub $(\hat{x}, \hat{y}, \hat{z})^T$.

The nominal inertial angular velocity of the system in its equilibrium orientation is given by

$$\underline{\Omega} = \Omega\, \hat{z} \tag{15}$$

---

*We anticipate that the preload effect is not functionally related to hub motions in Eq. (13); see Section II. B. 2.

Equation (14) can be rewritten as follows. The vector cross-product operation can be expressed as

$$\underline{a} \times \underline{b} = \{x\}^T \tilde{a} b = \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}^T \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{16}$$

Using this isomorphism, we can write the vector expression as

$$\{x\}^T \Big[m\,\{\ddot{\xi}\} + 2m[\tilde{\Omega}]\{\dot{\xi}\} + m\,[\tilde{\Omega}][\tilde{\Omega}]\{\xi\} + [K]\{\xi\} \Big] \tag{17}$$

where the difference in elastic forces has been replaced by $[K]\{\xi\}$.

1. CENTRIPETAL ACCELERATION EFFECTS

The contribution from the centripetal acceleration term is

$$m[\tilde{\Omega}][\tilde{\Omega}]\{\xi\} = m \begin{bmatrix} -\Omega^2 & 0 & 0 \\ 0 & -\Omega^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Big\{ \xi \Big\} \tag{18}$$

The deformation dependent terms collectively become

$$\begin{bmatrix} k_x - m\Omega^2 & 0 & 0 \\ 0 & k_y - m\Omega^2 & 0 \\ 0 & 0 & k_z \end{bmatrix} \Big\{ \xi \Big\} \tag{19}$$

where for convenience the stiffness matrix was assumed diagonal. The general effect of the centripetal acceleration is a reduction in stiffness or, analogously, a reduction in the unloaded frequency of oscillation of the structure. "Unloaded" denotes that the effects of preload, a deformation-dependent effect, have not been incorporated. The preload effect is still retained in the contact forces $\underline{F}'^{j}$. Preload effects and the elastic restoring effects collectively result in a stiffness matrix that, when suitably normalized, yields a set of loaded frequencies. These loaded frequencies are altered by the centripetal accelerations as discussed previously.

Terms in deformation rates also appear. Again, this is a spin-induced effect. The structure of the equations is altered by the presence of these terms from the form involving only mass and stiffness matrices to the more general homogeneous form

$$M\ddot{q} + G\dot{q} + Kq = 0 \tag{20}$$

The gyroscopic contribution $G\dot{q}$ has a unique structure in that the G matrix is skew symmetric. The role of the Coriolis acceleration can be important, but this discussion will be deferred until the remaining contact forces, called preload effects, are evaluated.

2. PRELOAD EFFECT

The remaining forces are evaluated for three special cases: a radially mounted subsystem, an anticantilever subsystem, and an intermediate combination of the two extremes. These configurations are illustrated in Figure 3. Anticipating the results, we find some marked differences between the configurations. The motivation for considering these three examples was provided in Ref. 5. The remaining contact forces in the radially mounted case are determined as follows. In its nominal equilibrium configuration, the massless

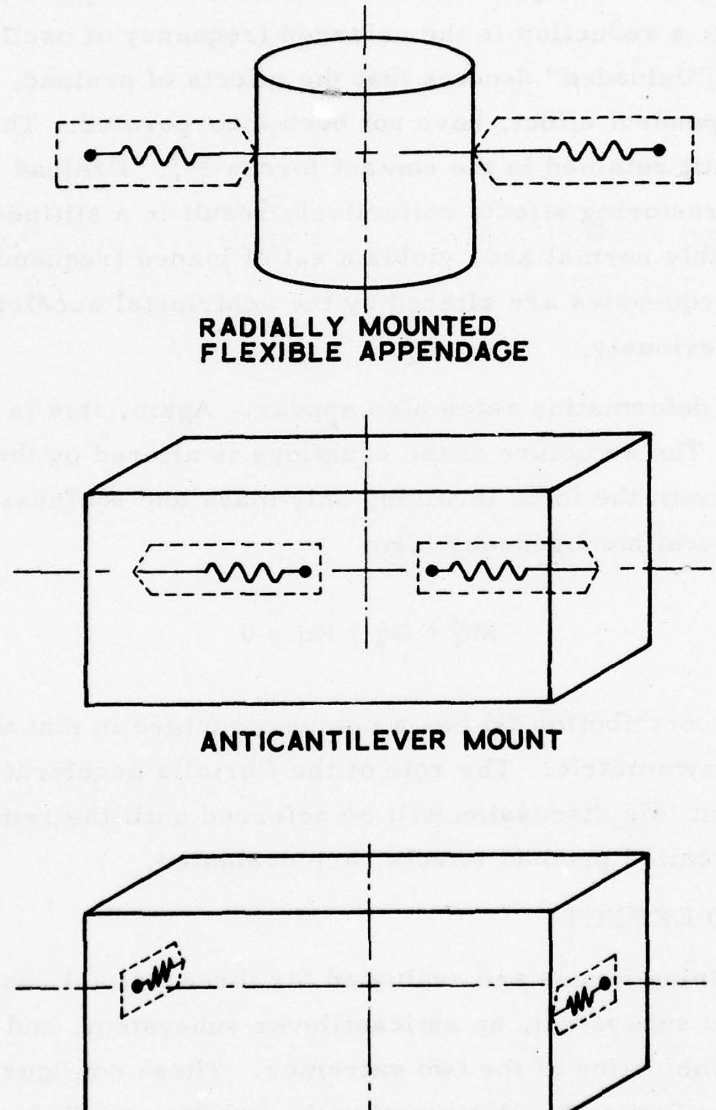

Figure 3. Appendage Subsystem Attachments

spring supporting the mass particle is in tension, the magnitude of which is $m\Omega^2 r_y$, where r_y could be further broken down into a nominal unloaded position plus a contribution induced by spin. For convenience of exposition only, assume that the mass flexural subsystem attachment point to the rigid hub is at the point 0. This configuration is illustrated in Figure 4.

To first approximation analysis, the remaining contact force $\underline{F}'^1$, resolved into the coordinate basis $[\hat{x}, \hat{y}, \hat{z}]$, is

$$\underline{F}'^1 = \{x\}^T \begin{Bmatrix} -m\,\Omega^2\,\delta_x \\ -m\,\Omega^2\,r_y \\ -m\,\Omega^2\,\delta_z \end{Bmatrix} \tag{21}$$

where the contribution from the preload effect tends to provide a restoring force and as such would positively augment the stiffness matrix [deformation-dependent terms in Eq. (12)].* A general stiffening effect is achieved through the preload effect for the radially mounted configuration. This effect is configuration dependent, as shown in the following example.

Consider the case of the <u>anticantilever</u> particle illustrated in Figure 5. Again, assume that the particle in its nominal equilibrium position is displaced to a distance r_y. For convenience it was assumed that the base of the spring is connected to the satellite at point P, a distance of $2r_y$ from the nominal center of mass. The interconnecting spring is in compression. A better pictorial representation of the nature of the anticantilever mount is presented in Figure 3.

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*The application of a transformation [Eq. (11)] does not alter this basic conclusion.

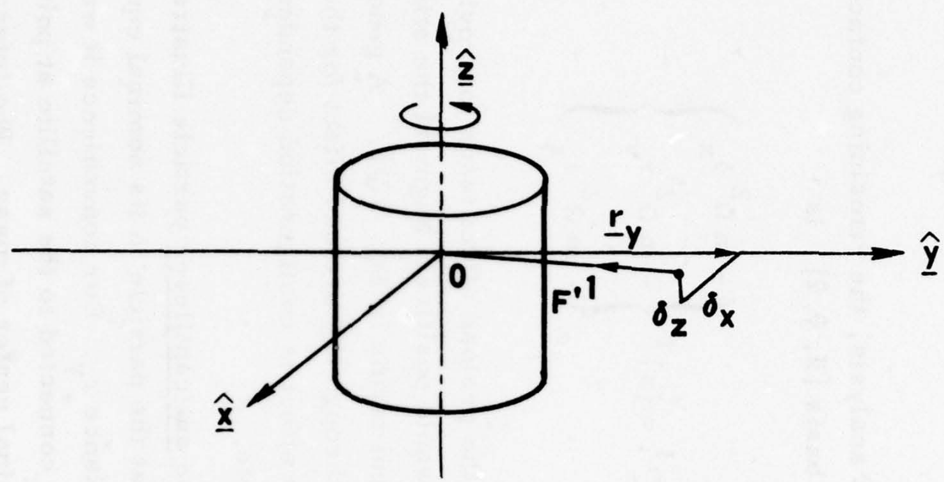


Figure 4. Preload Effect - Radially Mounted Configuration

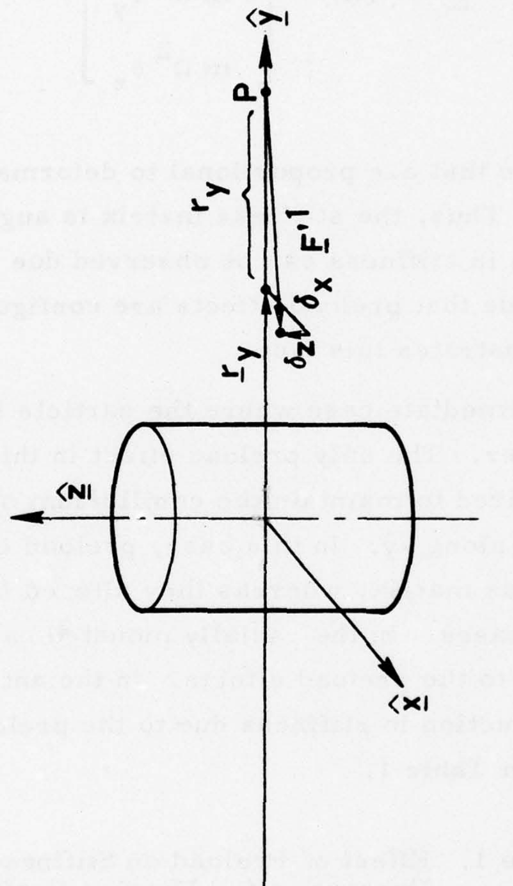


Figure 5. Preload Effect - Anticantilever Mount

The contact force, when resolved into the coordinate basis $[\hat{x}, \hat{y}, \hat{z}]^T$, is to first approximation

$$\underline{F'}^1 = [x]^T \begin{bmatrix} m\,\Omega^2\,\delta_x \\ -\,m\,\Omega^2\,r_y \\ m\,\Omega^2\,\delta_z \end{bmatrix} \tag{22}$$

The components of force that are proportional to deformations are not restoring in this case. Thus, the stiffness matrix is augmented such that a general reduction in stiffness can be observed due to preload effects. We can conclude that preload effects are configuration dependent. One final example reillustrates this fact.

Consider the intermediate case where the particle is neither radially mounted or anticantilever. The only preload effect in this instance is the steady-state force required to maintain the equilibrium orientation of the particle and is directed along $-\hat{y}$. In this case, preload effects do not contribute to the stiffness matrix, whereas they altered the stiffness matrix in the previous cases. In the radially mounted case, there was a general stiffening due to the preload effects. In the anticantilever case, there was a general reduction in stiffness due to the preload effects. These facts are summarized in Table 1.

Table 1. Effect of Preload on Stiffness Properties for Varying Configurations

Configuration	Qualitative Effect on Stiffness	Quantitative Effect on Stiffness[a] (Loaded Frequency Squared)
Radially Mounted	Increased stiffness	Eq. (21); $k/m + \Omega^2$
Anticantilever	Reduced stiffness	Eq. (22); $k/m - \Omega^2$
Neutral	No effect	

[a]Unloaded frequency $(k/m)^{1/2}$, when augmented by preload effect, is defined as loaded frequency.

Collectively, the effects of preload, nominal elastic restoring force effects, and centripetal accelerations are summarized in Table 2.

Table 2. Summary of Effects of Preload and Centripetal Acceleration on Stiffness Matrix[a]

$\left(\begin{matrix}\text{Loaded}\\\text{Frequency}\end{matrix}\right)^2$	Radial	Anticantilever	Neutral
σ_x^2	$(\frac{k_x}{m} + \Omega^2) - \Omega^2$	$(\frac{k_x}{m} - \Omega^2) - \Omega^2$	$(\frac{k_x}{m}) - \Omega^2$
σ_y^2	$(\frac{k_y}{m}) - \Omega^2$	$(\frac{k_y}{m}) - \Omega^2$	$(\frac{k_y}{m}) - \Omega^2$
σ_z^2	$(\frac{k_z}{m} + \Omega^2)$	$(\frac{k_z}{m} - \Omega^2)$	$(\frac{k_z}{m})$

[a]Loaded frequencies resulting from collective effects of elastic restoring forces and preload effects are in parentheses. Contributions from centripetal acceleration effects appear outside the parentheses.

3. CORIOLIS EFFECTS

We pointed out that terms in deformation rates appeared in Eq. (14) and that they were clearly a spin-induced effect. The basic structure of the equations is altered by the presence of these terms from the form involving only mass and stiffness matrices to the more general homogeneous form

$$M\ddot{q} + G\dot{q} + K\dot{q} = 0 \tag{23}$$

Further, the gyroscopic contribution $G\dot{q}$ has a unique structure in that the G matrix is skew symmetric. The role played by this effect is examined in the following paragraphs.

The homogeneous portion of Eq. (14) specialized for the case of radially mounted particles is

$$x^T \left[M\ddot{\xi} + 2M\tilde{\Omega}\,\dot{\xi} + K'\,\xi \right] = \underline{0} \tag{24}$$

where the matrix K' reflects the contributions from centripetal acceleration effects, preload, and normal elastic restoring forces. The scalar equivalent to Eq. (24) after dividing through by the mass is

$$
\begin{array}{llllll}
\text{Transverse} & \ddot{\xi}_x & -2\Omega\,\dot{\xi}_y & -\Omega^2\,\xi_x & +\Omega^2\,\xi_x & +\frac{k_x}{m}\,\xi_x = 0 \\
\text{to Spin} & \ddot{\xi}_y & +2\Omega\,\dot{\xi}_x & -\Omega^2\,\xi_y & & +\frac{k_y}{m}\,\xi_y = 0 \qquad (25) \\
\text{Spin} & \ddot{\xi}_z & & & +\Omega^2\,\xi_z & +\frac{k_z}{m}\,\xi_z = 0 \\
 & & \text{Coriolis} & \text{Centripetal} & \text{Preload} & \text{Elastic}
\end{array}
$$

The equations of deformation decouple into equations transverse to the spin direction and along the spin direction. Consider the equations of motion transverse to the spin direction (out-of-plane)

$$\ddot{\xi}_x - 2\Omega\,\dot{\xi}_y + \sigma_x^2\,\xi_x = 0$$

$$\ddot{\xi}_y + 2\Omega\,\dot{\xi}_x + \sigma_y^2\,\xi_y = 0 \tag{26}$$

where

$$\sigma_x^2 = \frac{k_x}{m} \text{ and } \sigma_y^2 = \frac{k_y}{m} - \Omega^2$$

If the hub were vanishingly small, then the frequencies considered here would be system frequencies. If we include the effects of the hub, the result of the Coriolis coupling effect might be altered. With this restriction, if we transform Eq. (26) into the S domain and evaluate the characteristic equation, we have

$$(S^2 + \sigma_x^2)(S^2 + \sigma_y^2) + \underbrace{4\Omega^2 S^2}_{\substack{\text{Coriolis} \\ \text{Coupling}}} = 0 \tag{27}$$

$$1 + \frac{K S^2}{(S^2 + \sigma_x^2)(S^2 + \sigma_y^2)} = 0 \tag{28}$$

In Figure 6 the poles represent the locations of frequencies σ_x and σ_y. They are located on the complex axis since we assume there is no damping in the system. The Coriolis coupling effects are represented by the factor K. The inclusion of Coriolis coupling can alter the system frequencies but does not alter the fact that the "system" roots, irrespective of gyroscopic coupling, must be on the imaginary axis in the event of no damping. The net effect is a further separation of the loaded frequencies. The highest root is pulled still higher, while the lowest root is further pulled to a lower frequency. Our intuition can be important in assessing the impact of this coupling effect. In the case of a spinning helicopter blade, we might assume the blade would be nearly inextensible radially. The loaded frequencies of the substructure would, in this case, be essentially unaltered by the Coriolis coupling effect ($\sigma_y^2 \gg \Omega^2$). In general, the importance of the Coriolis coupling effect must be assessed in each physical application and should not be casually dismissed.

$$1 + \frac{KS^2}{(S^2 + \sigma_x^2)(S^2 + \sigma_y^2)} = 0$$

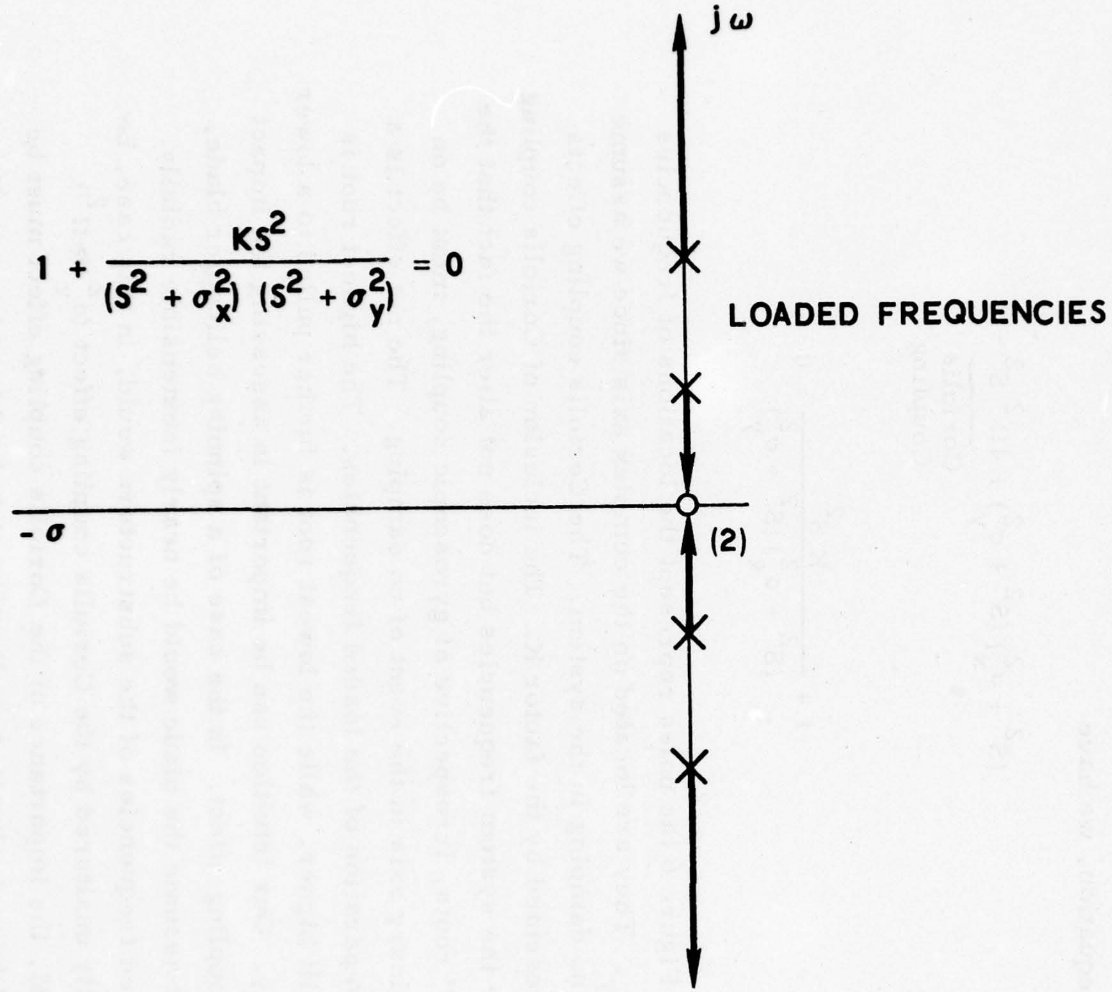


Figure 6. Coriolis Coupling Effect

C. COMPOSITE EQUATIONS OF MOTION

We have concentrated primarily on the development of equations of motion for the flexural subsystem. To complete the system description, we must augment this subset with equations of motion for the entire spacecraft.

This general approach has been delineated in Ref. 1 for more general dynamical systems than those presently under consideration. In particular, the general structure of equations is developed for the following:

1. Flexural appendages attached directly to a rigid hub
2. Flexural appendages engaged in spin
3. Systems incorporating discrete dampers and control system nonlinearities

The general structure of the equations developed in Ref. 1 could be used as a starting point for the simple example described here. The approach adopted here involves a development of composite equations of motion from first principles and parallels to some extent the development of equations in Ref. 1. However, the equation development is specialized to the example. Hopefully, this might serve as an introduction to the more comprehensive development.

In the following paragraphs, the composite equations of motion are developed. Since our aim is the development of expressions suitable for linearized stability analysis, a variational approach is adopted and first order approximation expressions are derived.

The total angular momentum of the composite system referenced to the composite center of mass is defined as

$$\underline{H}^c = \int \underline{\rho} \times \dot{\underline{\rho}}\, dm \tag{29}$$

where the vector $\underline{\rho}$ emanates from the instantaneous center of mass and terminates at a differential mass particle dm. The operator $(\dot{\ })$ is understood as an inertial time derivative, whereas the operator $(\mathring{\ })$ denotes a time derivative with respect to a designated reference frame. $\underline{H}^c$ can alternately be expressed as[6]

$$\underline{H}^c = {}^oJ \cdot \underline{\omega} + m_T \dot{\underline{\rho}}_o \times \underline{\rho}_o + \int \underline{r} \times \mathring{\underline{r}}\, dm \tag{30}$$

where ${}^oJ$ is the total system inertia dyadic referenced to point 0, $\underline{\omega}$ is the inertial angular velocity of a basis fixed in the spinner (for the example considered here), $\underline{\rho}_o$ is the location of point 0 referenced to the instantaneous ccm, and the vector $\underline{r}$ emanates from point 0 and terminates at a differential mass element.

From the rotational equivalent of Newton's second law of motion, we have

$$\frac{{}^N d}{dt} \underline{H}^c = \underline{T}^c \tag{31}$$

where $\underline{T}^c$ is a vector of all external torques referenced to the center of mass of the composite system. The inertial time derivative of the total system angular momentum can be expressed as

$$\begin{aligned} \frac{{}^N d}{dt} \underline{H}^c = {}^oJ \cdot \dot{\underline{\omega}} + (\underline{\omega} \times {}^oJ \cdot \underline{\omega}) + {}^o\mathring{J} \cdot \underline{\omega} \\ + \ddot{\underline{\rho}}_o \times \underline{\rho}_o m_T + \int \underline{r} \times \overset{\circ\circ}{\underline{r}}\, dm \\ + \underline{\omega} \times \int (\underline{r} \times \mathring{\underline{r}})\, dm \end{aligned} \tag{32}$$

The terms in Eq. (32) will be developed.

[6] R. E. Roberson, Dynamics and Control of Rotating Bodies, New York, Academic Press, to be published.

The integral $\int \underline{r} \times \overset{\circ}{\underline{r}}\, dm$, which is a relative angular momentum, can be evaluated as follows. For the case of particle masses illustrated in Figure 2, we have

$$\int \underline{r} \times \overset{\circ}{\underline{r}}\, dm = \sum_i \int_i (\underline{r}_i + \underline{\delta}_i) \times \overset{\circ}{\underline{\delta}}_i \, dm \tag{33}$$

or Eq. (33) can be alternatively expressed as

$$\int \underline{r} \times \overset{\circ}{\underline{r}}\, dm = m \sum_i (\underline{r}_i + \underline{\delta}_i) \times \overset{\circ}{\underline{\delta}}_i \tag{34}$$

The derivative of Eq. (34) with respect to a basis fixed in the hub is

$$\frac{{}^{P}d}{dt} \int \underline{r} \times \overset{\circ}{\underline{r}}\, dm = \int \underline{r} \times \overset{\circ\circ}{\underline{r}}\, dm = m \sum_i (\underline{r}_i + \underline{\delta}_i) \times \overset{\circ\circ}{\underline{\delta}}_i \tag{35}$$

The vector $\underline{\rho}_o$ can be expressed in terms of deformation of the particle masses as [Eq. (4)]

$$\underline{\rho}_o = - \frac{m}{m_T} \sum_i \underline{\delta}_i \tag{36}$$

The second inertial time derivative of Eq. (36) is

$$\ddot{\underline{\rho}}_o = - \frac{m}{m_T} \sum_i \Big[\overset{\circ\circ}{\underline{\delta}}_i + 2\, \underline{\omega} \times \overset{\circ}{\underline{\delta}}_i + \underline{\omega} \times (\underline{\omega} \times \underline{\delta}_i) + \dot{\underline{\omega}} \times \underline{\delta}_i \Big] \tag{37}$$

The vector cross-product $m_T \ddot{\underline{\rho}}_o \times \underline{\rho}_o$ is functionally dependent on products of $\underline{\delta}_i$ and its derivatives; as such it can be neglected in first approximation analysis.

The composite inertia dyadic referenced to the base point O can be expressed as

$$
\begin{aligned}
{}^{o}\mathbf{J} = \mathbf{I}^{o} + \Sigma\, m_i \,[\,(\underline{r}_i + \underline{\delta}_i)\cdot(\underline{r}_i + \underline{\delta}_i)\,\mathbf{E} \\
- (\underline{r}_i + \underline{\delta}_i)\,(\underline{r}_i + \underline{\delta}_i)\,]
\end{aligned}
\tag{38}
$$

where $\mathbf{I}^{o}$ represents the inertia of the hub referenced to point 0 and the summation represents the contribution of the flexural subsystem to the total system inertia referenced to point 0. The rotary inertia of the particle is assumed negligible. Equation (38) can be alternately expressed as

$$
{}^{o}\mathbf{J} = \{x\}^T\,[\,I^{o} - m \sum_i \widetilde{(r_i + \delta_i)}\,\widetilde{(r_i + \delta_i)}\,]\,\{x\} \tag{39}
$$

where the operator (~) represents a skew symmetric operation on the components of a (3 x 1) column matrix to yield

$$
\widetilde{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \tag{40}
$$

The elements of the (3 x 1) column matrix $\{x\}$ are unit vectors. A first approximation expression to Eq. (38) is

$$
\begin{aligned}
{}^{o}\mathbf{J} &\cong \{x^T\}[I^o - m\sum_i \tilde{r}_i\tilde{r}_i - m\sum_i(\tilde{r}_i\tilde{\delta}_i + \tilde{\delta}_i\tilde{r}_i)]\{x\} \\
&= \{x^T\}[I^* - m\sum_i(\tilde{r}_i\tilde{\delta}_i + \tilde{\delta}_i\tilde{r}_i)]\{x\}
\end{aligned} \tag{41}
$$

The derivative of Eq. (39) with respect to a platform fixed frame of reference is

$$
\begin{aligned}
\frac{{}^{P}d}{dt}\,{}^{o}\mathbf{J} = \overset{o}{{}^{o}\mathbf{J}} = \{x\}^T[&- m\sum_i \tilde{\dot{\delta}}_i\,\widetilde{(r_i + \delta_i)} \\
&- m\sum_i \widetilde{(r_i + \delta_i)}\,\tilde{\dot{\delta}}_i]\{x\}
\end{aligned} \tag{42}
$$

where differentiation has been carried out term-wise on only deformation-dependent terms.

Substituting Eqs. (34) and (35) into Eq. (32) and rewriting, we have, to first approximation for the case of no external torque,

$$
\begin{aligned}
&{}^{o}\mathbf{J}\cdot\dot{\underline{\omega}} + \underline{\omega}\times{}^{o}\mathbf{J}\cdot\underline{\omega} + \overset{o}{{}^{o}\mathbf{J}}\cdot\underline{\omega} + \ddot{\underline{\rho}}_o\times\underline{\rho}_o m_T \\
&+ m\sum_i(\underline{r}_i + \underline{\delta}_i)\times\overset{oo}{\underline{\delta}}_i + \underline{\omega}\times[m\sum_i(\underline{r}_i + \underline{\delta}_i)\times\overset{o}{\underline{\delta}}_i] = \underline{0}
\end{aligned} \tag{43}
$$

Following the previously established pattern, consider the system to be participating in pure spin in its nominal equilibrium state. The inertial angular velocity of the system in its equilibrium state is

$$
\underline{\Omega} = \Omega\hat{\underline{z}} \tag{44}
$$

The inertial angular velocity of the basis fixed in the hub is represented as

$$\underline{\omega} = \underline{\Omega} + \underline{\delta\omega} \tag{45}$$

where the angular velocity chain rule was used. A first approximation expression is developed by substitution of Eqs. (41), (42), and (45) into Eq. (43) and truncation of terms involving products of deformation, deformation rates, variational angular velocities, and variational accelerations to yield

$$\begin{aligned} x^T [& I^* \delta\dot{\omega} + \widetilde{\delta\omega} I^* \Omega + \tilde{\Omega} I^* \delta\omega \\ & + \tilde{\Omega} (I^* - m \Sigma \tilde{\delta}_i \tilde{r}_i - m \Sigma \tilde{r}_i \tilde{\delta}_i) \Omega \\ & + \quad - m \Sigma \tilde{\dot{\delta}}_i r_i \Omega - m \Sigma \tilde{r}_i \tilde{\dot{\delta}}_i \Omega \\ & + m \Sigma \tilde{r}_i \ddot{\delta}_i + \tilde{\Omega} m \Sigma \tilde{r}_i \dot{\delta}_i] = 0 \end{aligned} \tag{46}$$

With some specialization to a diagonal inertia matrix, the corresponding scalar equations coordinatized in a platform-fixed basis are

$$\begin{aligned} & A\, \delta\dot{\omega}_x + \Omega (C-B)\, \delta\omega_y + \Sigma m_i r_y^i (\ddot{\delta}_z^i + \Omega^2 \delta_z^i) = 0 \\ & B\, \delta\dot{\omega}_y + \Omega (A-C)\, \delta\omega_x = 0 \\ & C\, \delta\dot{\omega}_z + 2 m \Sigma \dot{\delta}_y^i \Omega r_y^i - m \Sigma r_y^i \ddot{\delta}_x^i = 0 \end{aligned} \tag{47}$$

Assume the particles are symmetrically located about the hub so

$$r_y^1 = -r_y^2 = \Gamma_y \tag{48}$$

Applying the transformations [Eq. (11)], we can express Eq. (47) as

$$\text{Out of Plane}\quad\begin{cases} A\ \delta\dot{\omega}_x + \Omega\,(C-B)\,\delta\omega_y + m\Gamma_y[\ddot{\xi}_z + \Omega^2\,\xi_z] = 0 \\ B\ \delta\dot{\omega}_y + \Omega\,(A-C)\,\delta\omega_x = 0 \end{cases}$$

$$\text{In Plane}\quad\begin{cases} C\ \delta\dot{\omega}_z + 2\Omega\Gamma_y m\,\dot{\xi}_y - m\Gamma_y\ddot{\xi}_x = 0 \end{cases} \tag{49}$$

The dependent variable η, which is associated with a symmetric mode, is not present in the rotational equations. Earlier we observed that rotations of the hub did not couple into the symmetric mode shape. The excitation of the symmetric mode shape does not influence the hub rotational motions, nor do hub rotational motions tend to excite symmetric modes of the structure. In addition, there is a decoupling between in- and out-of-plane motions.

D. SYSTEM EQUATIONS

The system equations are comprised of the total composite equations of motion [Eq. (49)] and flexural subsystem equation (12) specialized for the radially mounted case, repeated here for convenience:

$$\text{Out of Plane}\quad\begin{cases} A\ \delta\dot{\omega}_x + \Omega\,(C-B)\,\delta\omega_y + m\,\Gamma_y\,(\ddot{\xi}_z + \Omega^2\xi_z) = 0 \\ B\ \delta\dot{\omega}_y + \Omega\,(A-C)\,\delta\omega_x = 0 \end{cases}$$

$$\text{In Plane}\quad\begin{cases} C\ \delta\dot{\omega}_z + 2\Omega\,\Gamma_y\,m\,\dot{\xi}_y - m\,\Gamma_y\,\ddot{\xi}_x = 0 \end{cases}$$

$$\ddot{\xi}_x - 2\Omega\,\dot{\xi}_y + k/m\,\xi_x - 2\delta\dot{\omega}_z\,\Gamma_y = 0$$

$$\ddot{\xi}_y + 2\Omega\,\dot{\xi}_x + (k/m - \Omega^2)\,\xi_y - 4\delta\omega_z\,\Omega\,\Gamma_y = 0$$

$$\ddot{\xi}_z + (\Omega^2 + k/m)\,\xi_z + 2\delta\dot{\omega}_x\,\Gamma_y + 2\delta\omega_y\,\Omega\,\Gamma_y = 0 \tag{50}$$

The equations describing wobble motions decouple as follows:

$$A\ \delta\dot{\omega}_x + \Omega\,(C-B)\ \delta\omega_y + m\,\Gamma_y\,(\ddot{\xi}_z + \Omega^2\,\xi_z) = 0$$

$$B\ \delta\dot{\omega}_y + \Omega\,(A-C)\ \delta\omega_x = 0$$

$$\ddot{\xi}_z + (\Omega^2 + k/m)\ \xi_z + 2\delta\dot{\omega}_x\,\Gamma_y + 2\,\delta\omega_y\,\Omega\,\Gamma_y = 0 \tag{51}$$

The additional equations descriptive of hub motions along the spin axis can be decoupled as follows:

$$C\ \delta\dot{\omega}_z + 2\,\Omega\,\Gamma_y\,m\,\dot{\xi}_y - m\,\Gamma_y\,\ddot{\xi}_x = 0$$

$$\ddot{\xi}_x - 2\Omega\,\dot{\xi}_y + k/m\ \xi_x - 2\delta\dot{\omega}_z\,\Gamma_y = 0$$

$$\ddot{\xi}_y + 2\Omega\,\dot{\xi}_x + (k/m - \Omega^2)\ \xi_y - 4\delta\omega_z\,\Omega\,\Gamma_y = 0 \tag{52}$$

III. STABILITY ANALYSIS

A stability analysis is provided for the simplified spacecraft system. Emphasis is on the development of literal stability criteria of value in preliminary design phases. Admittedly, some risk is associated with extrapolation of results for simplified dynamical systems to more complicated examples. The stability criteria can be of value in preliminary design phases and are not intended to supplant more detailed followup studies to verify ultimate system stability.

The results presented in this section parallel closely the work of P. Likins and F. Barbera.[5] Their work addressed the problem of developing literal attitude stability criteria for idealized spinning spacecraft. The stability criteria essentially represented a duplication of results from several prior studies involving considerably more sophisticated models as a point of departure. The essential results of that effort are described.

A. STABILITY ANALYSIS OF SIMPLE PARTICLE MODEL

The equations descriptive of the wobble motion for the simple particle model of Figure 1, provided in Section II [Eq. (51], are repeated here for convenience:*

$$A\,\delta\dot{\omega}_x - (B-C)\,\Omega\,\delta\omega_y + m\,\Gamma_y\,(\ddot{\xi}_z + \Omega^2\,\xi_z) = 0$$

$$B\,\delta\dot{\omega}_y + (A-C)\,\Omega\,\delta\omega_x = 0$$

$$\ddot{\xi}_z + 2\zeta\,\sigma\,\dot{\xi}_z + \sigma^2\,\xi_z + 2\delta\dot{\omega}_x\,\Gamma_y + 2\,\delta\omega_y\,\Omega\,\Gamma_y = 0 \qquad (53)$$

---

*An additional damping term has been included in the third equation (ζ is the damping ratio). σ represents the loaded frequency for the radially mounted subsystem.

A, B, and C represent total system inertias about $\hat{x}$, $\hat{y}$, and $\hat{z}$ respectively; ξ_z is the difference between the z axis deformation of the particles; and $\delta\omega_x$ and $\delta\omega_y$ are the angular velocity variational coordinates from nominal spin. The stability of that portion of the system descriptive of wobble motions can be obtained by examination of the characteristic equation using Routh-Hurwitz criteria, which yield both necessary and sufficient conditions.

The characteristic equation associated with Eqs. (53) can be expressed as

$$
\begin{aligned}
& S^4 + [\,2\zeta\,\sigma + 2\zeta\,\sigma\,K'\,]\;S^3 \\
& + [\,\alpha^2\,\beta^2 + \sigma^2 + K'\,(\sigma^2 - \Omega^2]\;S^2 \\
& + [\,2\zeta\,\sigma\,\alpha^2\,\beta^2 + 2\zeta\,\sigma\,K'\,\alpha^2]\;S \\
& + [\sigma^2\,\alpha^2\,\beta^2 + K'\,\alpha^2\,(\sigma^2 - \Omega^2)] = 0
\end{aligned}
\tag{54}
$$

where

$$K' = \frac{2m\,\Gamma_y^2}{A'}$$

$$\alpha^2 = \frac{\Omega^2\,(C' - A')}{B'}$$

$$\beta^2 = \frac{(C' - B')}{A'}$$

and A′, B′, and C′ represent core inertias. The characteristic equation is of the following form:

$$S^4 + p_3 S^3 + p_2 S^2 + p_1 S + p_0 = 0 \tag{55}$$

One form of the Routh-Hurwitz criteria yields the following necessary and sufficient conditions for asymptotic stability:

Criterion 1 $p_3 > 0$

Criterion 2 $p_1 > 0$

Criterion 3 $p_1 p_2 p_3 - p_1^2 - p_0 p_3^2 > 0$

Criterion 4 $p_0 > 0$

The results obtained by applying these criteria for the system under consideration are summarized in Table 3.

Criterion 4 can alternatively be expressed in terms of hub inertias B′, C′, as

$$\left(\frac{\sigma}{\Omega}\right)^2 > \frac{2m\,\Gamma_y^2}{2m\,\Gamma_y^2 + C' - B'} \tag{56}$$

The loaded frequency σ^2 for the case of the radially mounted particle, given in Table 1, is repeated here as

$$\underset{\text{Loaded Frequency}}{\sigma^2} = \underset{\text{Natural Frequency}}{k/m} + \underset{\text{Preload}}{\Omega^2} \tag{57}$$

Table 3. Necessary Conditions for Stability of Spacecraft Motions[a]

Criterion	Condition	Comments
1	$\zeta > 0$	Positive damping
2, 3	$C > A$ $C > B$	Spin about axis of maximum moment of inertia (system)
4	$\left(\frac{\sigma}{\Omega}\right)^2 > \frac{2m\,\Gamma_y^2}{C - B}$	Bounding criteria on loaded natural frequency in terms of spin frequency and system inertia properties

[a]The criteria are both necessary and sufficient conditions for asymptotic stability for equations associated with the wobble motions of the spacecraft. The conditions are only necessary conditions for asymptotic stability of equations of spacecraft.

and is augmented positively by the effects of preload. If the hub spin inertia exceeds the transverse inertia, then the inequality [Eq. (56)] is trivially satisfied and furthermore is independent of the amount of flexibility or the magnitude of spin Ω.

The dynamical model and the associated stability criteria developed for the wobble motion of the simplified spacecraft system are sufficiently general to permit examination of alternative ways of attaching the flexural subsystem. We observed in Section II that the anticantilever-mounted substructure experienced a reduction in its natural frequency due to the preload effect, whereas the intermediate case did not experience a pulling of its natural frequency due to preload effects. If we assume that the hub inertias

$$C' = B'$$

then the anticantilever-mounted subsystem would lead to more stringent criteria on the unloaded structural subsystem to ensure that system wobble motions are stable. We can easily see this by substituting the expression for the loaded frequency of the anticantilever subsystem

$$\sigma^2 = k/m - \Omega^2 \tag{58}$$

into the inequality [Eq. (56)] and observing that stability is ensured only if the unloaded structural frequency squared is at least twice as great as the spin frequency squared ($k/m > 2\Omega^2$). The orthogonally mounted subsystem requires that the unloaded structural frequency be greater than the spin frequency $[(k/m)^{1/2} > \Omega]$.

B. COMPARISON WITH PREVIOUS STUDIES

The literal stability criteria presented have been developed for a simple particle construct that is supposedly representative of a more general flexural subsystem. The fidelity of such criteria in predicting stability requirements for more complex spacecraft might well be questioned. Likins and Barbera provided comparisons with stability analysis results published in the literature for a specific class of spacecraft exhibiting flexural booms mounted radially outward from a rigid core.[7-10] With proper interpretation of results,

---

[7] J. E. Rakowski and M. L. Renard, "A Study of the Nutational Behavior of a Flexible Spinning Satellite Using Natural Frequencies and Modes of the Rotating Structure," AAS/AIAA Astrodynamics Conference, August 19-21, 1970, Santa Barbara, California.

[8] F. R. Vigneron, "Stability of a Freely Spinning Satellite of Crossed-Dipole Configuration," Canadian Aeronautics and Space Institute Transactions, 3, 1: 8-19, March 1970.

[9] T. W. Flatley, Attitude Stability of a Class of Partially Flexible Spinning Satellites, NASA, Goddard Space Flight Center, Technical Note No. D-5268, August 1969.

[10] L. Meirovitch and R. A. Calico, "The Stability of Motion of Force-Free Spinning Satellites with Flexible Appendages," AAS/AIAA Astrodynamics Specialists Conference, August 17-19, 1971, Fort Lauderdale, Florida.

the literal stability criterion developed for the simple particle idealization and the stability criterion developed for models which were more complex in idealization give surprisingly close agreement. The details of these comparisons are discussed.

Rakowski and Renard[7] studied the nutational behavior of a torque-free spinning satellite composed of a rigid hub and flexible appendages in an equatorial plane. The vehicle studied is referred to in the literature as a crossed dipole configuration. The flexible appendages are two pairs of uniform booms located along the transverse axes of principal inertia of the hub. Two separate classes of antenna flexure considered were equatorial plane and meridian plane motions. The following limiting assumptions were made: small displacement vibrations, nearly constant spin speed of the satellite, and small products of the transverse angular rates as compared to the square of the boom natural frequency. These assumptions validate the use of previously determined natural frequency and mode shape data that reflect the effects of spin. The fundamental mode of vibration was assumed to predominate. The justification for the truncation of higher modes was that the rigid body forcing occurred at a frequency closest to the loaded lowest natural frequency of the booms. The response in the higher modes to that excitation was deemed small. The stability of this configuration for a variety of system parameter variations was assessed by numerical integration of the full nonlinear equations of motion. Stability boundaries were developed in plots of normalized inertia properties for different values of a nondimensional parameter that reflects a ratio of centrifugal-to-elastic forces. These parameters are defined in terms of variables used herein. Let the normalized inertia parameters be defined as

$$\Gamma \triangleq \frac{3\,I_B}{C'} \tag{59}$$

$$K_p \triangleq \frac{C'}{B'} \tag{60}$$

where I_B is the inertia of a uniformly distributed beam about the core. In the simple particle model, the analogy would be

$$I_B = m \Gamma_y^2 \tag{61}$$

The nondimensional ratio $\overline{\lambda}$ is defined as

$$\overline{\lambda} \triangleq \frac{\Omega^2}{EI/\rho \ell^4} \tag{62}$$

A necessary condition for stability* for a particle mass approximation to a crossed dipole configuration is

$$\left(\frac{\omega_i}{\Omega} \right)^2 > \frac{2m_i \, d_i^2}{2m_i \, d_i^2 + C' - I_i} \tag{63}$$

where ω_i is the loaded frequency associated with the i^{th} axis, d_i is the nominal displacement of the i^{th} particle from the axis of spin, and I is the inertia of the hub about the i^{th} axis. The necessary conditions for stability for the crossed dipole configuration can be broken down into two separate criteria that are nearly identical with the criteria developed for the case of a rigid hub and a single pair of radially mounted booms.

*Developed in Ref. 5 and summarized in the Appendix

The only other identification we must make to complete the correspondence of the literal stability criteria results with the work of Rakowski and Renard is the loaded frequency. The loaded frequency of a uniform boom is[5, 8]

$$\omega^2 = \hat{\sigma}^2 + 1.193\,\Omega^2 \tag{64}$$

where $\hat{\sigma}$ is the lowest unloaded natural frequency of a uniform boom* and is given as[11]

$$\hat{\sigma} = (3.515)\left(\frac{EI}{\rho \ell^4}\right)^{1/2} \tag{65}$$

By comparison, the loaded frequency of a simple particle analogy is

$$\omega^2 = \hat{\sigma}^2 + \Omega^2 \tag{66}$$

Thus, despite a significant difference in models, the loaded frequencies are nearly the same.

The simple particle stability criteria, with minor interpretation, can be reexpressed in terms of the nondimensionalized parameters used in Rakowski and Renard's work. The original necessary condition for stability can be reexpressed as*

---

*Likins and Barbera used a loaded frequency of $\omega^2 = \hat{\sigma}^2 + 1.2\,\Omega^2$, which is generated for a massless uniform beam with particle tip mass.

[11] W. C. Hurty and M. F. Rubinstein, Dynamics of Structures, New Jersey, Prentice-Hall, 1964.

$$\frac{3.515^2}{\Omega^2}\frac{EI}{p\ell^4} + 1.193 > \frac{\frac{2}{3}\Gamma}{\frac{2}{3}\Gamma + 1 - (1/K_p)} \tag{67}$$

or

$$\frac{12.4}{\bar{\lambda}} + 1.193 > \frac{\frac{2}{3}\Gamma}{\frac{2}{3}\Gamma + 1 - (1/K_p)} \tag{68}$$

Rakowski and Renard's results are reproduced in Figure 7. The results of Barbera and Likins, which amount to a literal stability criterion developed for a simple particle analogy, compare very favorably after some minor reinterpretation outlined above.

Additional comparisons have been provided in Ref. 5 with other analyses.[8, 10] These comparisons are not reiterated here. All these comparisons substantiate the fact that the results developed for the particle analogy, with some minor reinterpretation, can serve as a useful starting point in preliminary design work. As in most studies, there are certain limiting-case situations for which the criteria might be misleading: for example, high spin-to-stiffness ratios. At the limiting case of high spin-to-stiffness ratios, we would expect that, as the necessary condition for stability, the hub spin inertia must exceed a transverse inertia of the hub.* And yet, if we use the approximation for the loaded frequency $\omega^2 = (\hat{\sigma}^2 + 1.2\,\Omega^2)$, then the requirement $(C' > B')$ does not emerge from the stability inequality. At high spin-to-stiffness ratios, the loaded frequency is close to the loaded frequency predicted for the simple particle analogy $(\omega^2 = \hat{\sigma}^2 + \Omega^2)$. The ratio of loaded frequencies to spin frequency for a uniform massless beam with a particle tip mass is analytically given as[5]

*This result can be verified in Ref. 9.

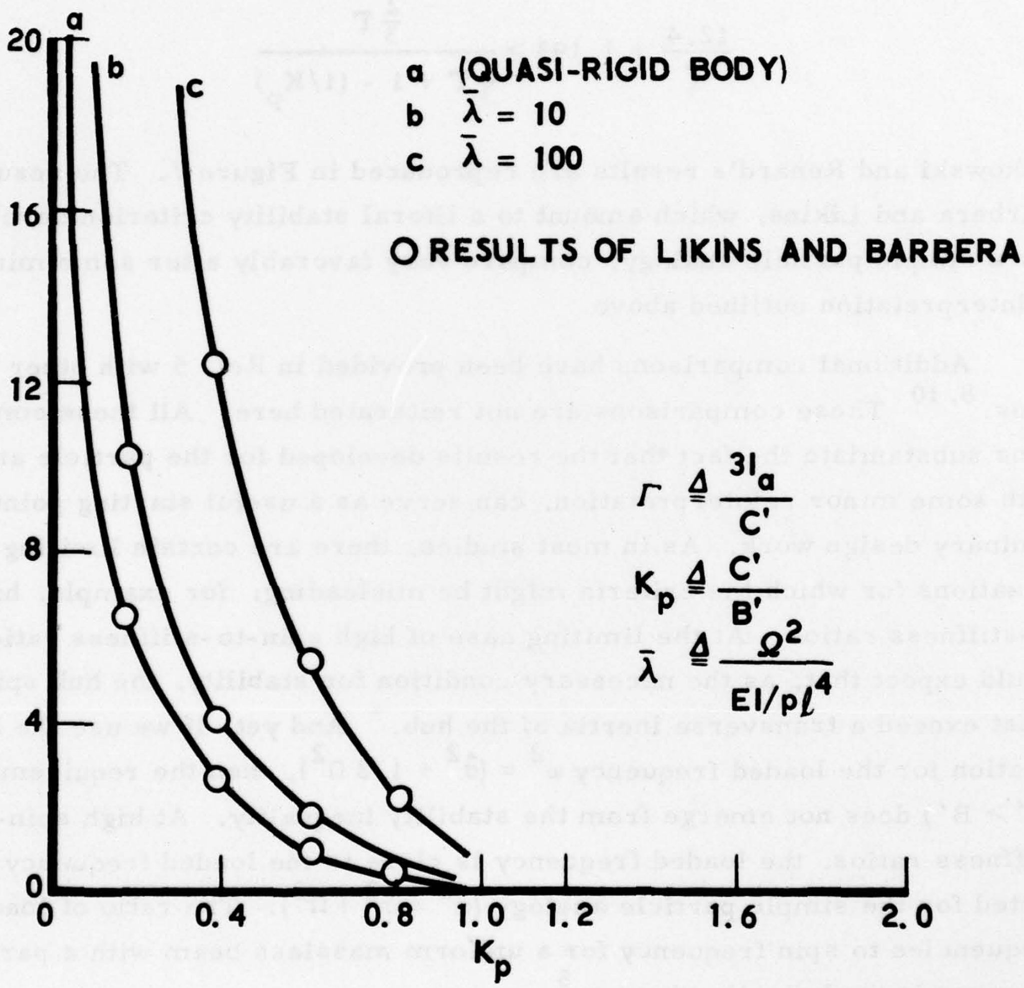


Figure 7. Comparison with Previous Results

$$\left(\frac{\omega}{\Omega}\right)^2 = \frac{1}{1 - \left| \tanh\left[(3)^{1/2} (\Omega/\hat{\sigma})\right] / \left[(3)^{1/2} (\Omega/\hat{\sigma})\right] \right|} \tag{69}$$

and can be verified to approach a value of 1 as $\Omega/\hat{\sigma}$ approaches infinity. Thus, the discrepancy arises in approximating the loaded frequency of the uniform massless beam with particle tip mass as

$$\left(\frac{\omega}{\Omega}\right)^2 = \left(\frac{\hat{\sigma}}{\Omega}\right)^2 + 1.2 \tag{70}$$

as opposed to its true value at high spin-to-stiffness ratios, which in the limiting case is one ($\frac{\omega}{\Omega} \rightarrow 1$ as $\frac{\Omega}{\hat{\sigma}} \rightarrow \infty$).

C. STABILITY CRITERIA EXTENSIONS

To date, literal stability criteria have been developed for simplified particle mass idealizations of a spacecraft. Literal stability criteria have been developed for more general structures configured such that the appendage particles in their undeformed state lie in a plane perpendicular to the principal axis of spin and passing through the system center of mass.[5] These developments are algebraically cumbersome and do not permit the clear interpretations provided in the simpler examples. As a result, only the basic outline of the approach taken is provided, along with the major results of the study.[5] The reader is referred to the original study for a more complete explanation.

The equations of motion for the appendage are developed for the i^{th} particle of the appendages idealized as described above. The equation development is nearly identical to the development in Section II. An equally acceptable starting point might be simplification of the more general structure of equations provided in Ref. 1. Further assuming that the contact forces resolved in the transverse plane are independent of z axis deformations or their rates or of contact forces resolved along the z axis deformations or their rates, we can verify the wobble axis motions do decouple from in-plane

motions, as observed previously for the limiting physical example. The resulting equations contain the dependent variables u_{zi}, the z axis deformation of the i^{th} particle, and the two transverse variational angular velocities ω_x and ω_y. Collectively, these generic equations for the appendage subsystem are given in Eq. (71) in a matrix format with the vector q identified as an N vector of z axis displacements. The appendage subsystem equations reduce to

$$M\ddot{q} + Kq = M\Gamma_x \ell \ (\dot{\omega}_y - \Omega\omega_x) - M\Gamma_y \ell \ (\dot{\omega}_x + \Omega\omega_y) \tag{71}$$

where the contact forces have been replaced by their matrix counterparts, and where

$$M = \begin{bmatrix} m_1 \quad \dfrac{m_1^2}{M_T} & \cdots & \dfrac{-m_1 m_n}{M_T} \\ \vdots & \ddots & \\ \dfrac{-m_1 m_n}{M_T} & & m_n - \dfrac{m_n^2}{M_T} \end{bmatrix}$$

$$\Gamma_x = \begin{bmatrix} \Gamma_x^1 & \cdots & 0 \\ \vdots & \Gamma_x^2 & \\ 0 & & \ddots \quad \Gamma_x^N \end{bmatrix}$$

$$\ell = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \qquad (N \times 1)$$

The wobble motion portion of the system equations reduces to

$$A\delta\dot{\omega}_x - \Omega\delta\omega_y(B-C) + \Omega^2 (M\Gamma_y \ell)^T q + (M\Gamma_y \ell)^T \ddot{q} = 0$$

$$B\delta\dot{\omega}_y + \Omega\delta\omega_x(A-C) - \Omega^2 (M\Gamma_x \ell)^T q - (M\Gamma_x \ell)^T \ddot{q} = 0 \tag{72}$$

The column matrix q will be replaced by $\Phi\eta$, where Φ is a matrix of eigenvectors obtained from the eigenvalue problem associated with the homogeneous matrix equation

$$M\ddot{q} + Kq = 0 \tag{73}$$

With appropriate normalization of Φ such that

$$\Phi^T M\Phi = E \quad (N \times N) \tag{74}$$

and with the addition of modal damping, the appendage equations reduce to*

$$\ddot{\eta} + 2\zeta\,\omega\dot{\eta} + \omega^2\eta = \delta_x(\delta\dot{\omega}_y - \Omega\delta\omega_x) - \delta_y(\delta\dot{\omega}_x + \Omega\delta\omega_y) \tag{75}$$

---

*At the risk of duplicating nomenclature, the variable η was selected to represent a modal deformation. The same variable was used earlier in the text, the definition of which appears in Eq. (11).

where

$$\delta_x = \Phi^T M \Gamma_x \ell \qquad (N \times 1)$$

$$\delta_y = \Phi^T M \Gamma_y \ell \qquad (N \times 1)$$

and the equations descriptive of wobble motion reduce to

$$A\delta\dot{\omega}_x - \Omega\delta\omega_y (B-C) + \Omega^2 \delta_y^T \eta + \delta_y^T \ddot{\eta} = 0$$

$$B\delta\dot{\omega}_y + \Omega\delta\omega_x (A-C) - \Omega^2 \delta_x^T \eta - \delta_x^T \ddot{\eta} = 0$$

These equations are subsequently specialized by truncating to one modal coordinate. The $(N \times 1)$ vectors δ_x and δ_y reduce to scalars designated as δ_{x_1} and δ_{y_1}. With the corresponding reduction in dimensionality, literal stability criteria can be developed by Routh-Hurwitz analysis. The necessary and sufficient conditions for asymptotic stability of the wobble motion are given in Table 4, based on the indicated restrictions.

Table 4. Necessary and Sufficient Conditions for Stability of Wobble Motion for Single Mode Case

$$\left.\begin{array}{l} C > A \\ C > B \end{array}\right\} \text{Major axis spin requirement}$$

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\delta_{x_1}^2 (C-B) + \delta_{y_1}^2 (C-A)}{(C-A)(C-B)}$$

In the previous analysis, one restriction was that all modes were truncated with the exception of a single modal frequency. Can satisfaction of the inequality in Table 4 for each mode individually ensure stability for the system in which all modes are present? Necessary conditions can be developed using Routh-Hurwitz analysis for the multimode case. The development can be algebraically encumbered if we attempt to develop all the necessary and sufficient conditions for stability. However, by concentrating solely on a single inequality of the Routh-Hurwitz criteria, necessary conditions can be identified. In particular, the requirement that the constant term from the characteristic equation be greater than zero [Eq. (55)] yields the criteria indicated in Table 5 for the cases indicated.

Table 5. Necessary Conditions for Stability of Wobble Motion for Multimode Case

Case	Criterion
$\Gamma_x = 0$	$\frac{1}{\Omega^2} > \frac{\delta_y^T (\omega^2)^{-1} \delta_y}{C-B}$
$\Gamma_y = 0$	$\frac{1}{\Omega^2} > \frac{\delta_x^T (\omega^2)^{-1} \delta_x}{C-A}$
$\Gamma_x, \Gamma_y \neq 0$; $\left[\delta_x \delta_y^T - \delta_y \delta_x^T\right] = 0$	$\frac{1}{\Omega^2} > \frac{(C-A)\, \delta_y^T (\omega^2)^{-1} \delta_y + (C-B)\, \delta_x^T (\omega^2)^{-1} \delta_x}{(C-A)(C-B)}$

The subject of enhancing stability of a spinning flexible body through active control was briefly discussed in Ref. 5. The results achieved to date are summarized in the Appendix. This area in particular deserves considerable attention and will be the subject of much continuing investigation.

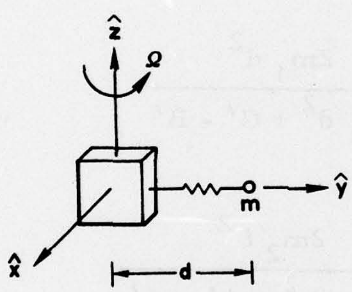
APPENDIX

SUMMARY OF STABILITY CRITERIA*

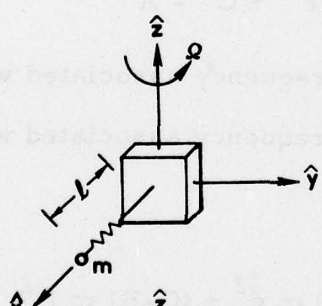
The following criteria have been formally established as necessary conditions** for asymptotic stability of spin.

C > A AND C > B IN ALL MODELS

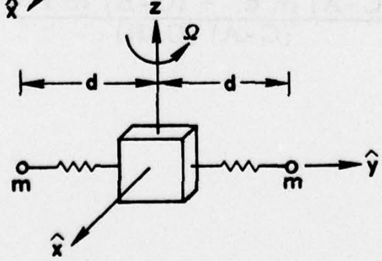
MODEL | CRITERION



$$\left(\frac{\omega}{\Omega}\right)^2 > \frac{m\,d^2}{(C-B)}$$



$$\left(\frac{\omega}{\Omega}\right)^2 > \frac{m\,\ell^2}{(C-A)}$$



$$\left(\frac{\omega}{\Omega}\right)^2 > \frac{2\,m\,d^2}{(C-B)}$$

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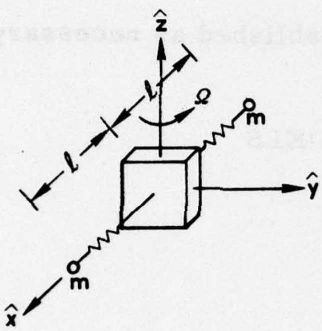
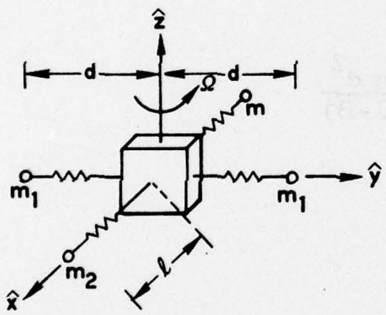
*Reproduced in its entirety from Ref. 5 by permission of the authors.

**In some cases sufficiency has also been proven formally.

MODEL | CRITERION



$$\left(\frac{\varepsilon}{\Omega}\right)^2 > \frac{2\, m\, \ell^2}{(C-A)}$$

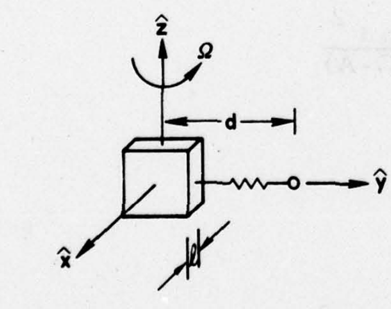


$$\left(\frac{\varepsilon}{\Omega}\right)^2 > \frac{2m_1\, d^2}{2m_1\, d^2 + C' - B'}$$

$$\left(\frac{\sigma}{\Omega}\right)^2 > \frac{2m_2\, \ell^2}{2\, m_2\, \ell^2 + C' - A'}$$

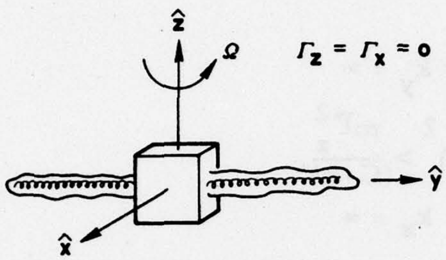
$\omega \sim$ loaded frequency associated with m_1

$\sigma \sim$ loaded frequency associated with m_2



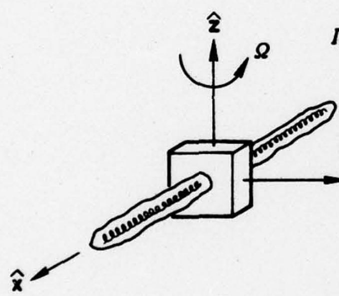
$$\left(\frac{\varepsilon}{\Omega}\right)^2 > \frac{(C-A)\, m\, d^2 + (C-B)\, m\, \ell^2}{(C-A)\,(C-B)}$$

MODEL	CRITERION



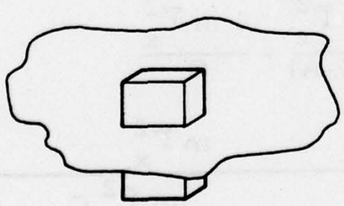
$\Gamma_z = \Gamma_x = 0$

$$\left(\frac{1}{\Omega}\right)^2 > \frac{\delta_y^T [\omega^2]^{-1} \delta_y}{(C-B)}$$



$\Gamma_z = \Gamma_y = 0$

$$\left(\frac{1}{\Omega}\right)^2 > \frac{\delta_x^T [\omega^2]^{-1} \delta_x}{(C-A)}$$



$\Gamma_z = 0$

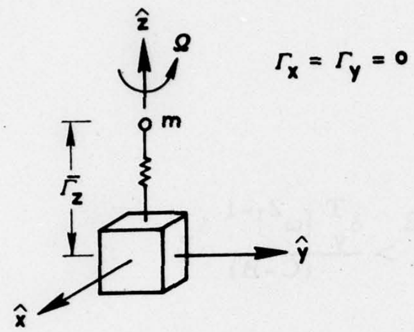
1^{st} Mode

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{(C-A)\,\delta_{y1}^2 + (C-B)\,\delta_{x1}^2}{(C-A)\,(C-B)}$$

N^{th} Mode $(\delta_x \delta_y^T - \delta_y \delta_x^T) = 0$

$$\left(\frac{1}{\Omega}\right) > \frac{(C-A)\,\delta_y^T [\omega^2]^{-1} \delta_y + (C-B)\,\delta_x^T [\omega^2]^{-1} \delta}{(C-A)\,(C-B)}$$

MODEL | CRITERION

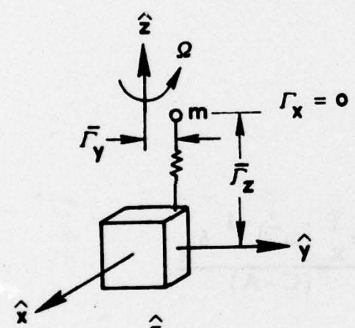


For $k_y = \infty$

$$\left(\frac{\omega_x}{\Omega}\right)^2 > \frac{m\bar{\Gamma}_z^2}{(C-A)}$$

For $k_x = \infty$

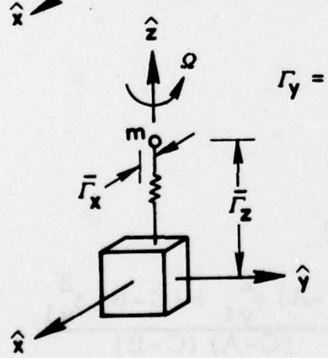
$$\left(\frac{\omega_y}{\Omega}\right)^2 > \frac{m\,\bar{\Gamma}_z^2}{(C-B)}$$



$k_y = \infty$

$$\left(\frac{\omega_x}{\Omega}\right)^2 > \frac{m\,\bar{\Gamma}_z^2}{(C-A)}$$

$$\left(\frac{\omega_z}{\Omega}\right)^2 > \frac{m\bar{\Gamma}_y^2}{(C-B)}$$



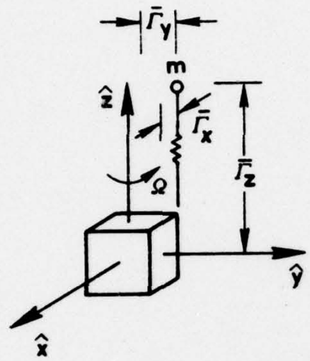
$k_y = \infty$

$$\left(\frac{\omega_x}{\Omega}\right)^2 > \frac{m\,\bar{\Gamma}_z^2}{(C-A)} - \frac{4\,m\,\bar{\Gamma}_x^2}{C}$$

$$\left(\frac{\omega_z}{\Omega}\right)^2 > \frac{m\,\bar{\Gamma}_x^2}{(C-A) - \dfrac{m\,\bar{\Gamma}_z^2\,C}{\left[C\left(\frac{\omega_x}{\Omega}\right)^2 + 4m\bar{\Gamma}_x^2\right]}}$$

MODEL

CRITERION



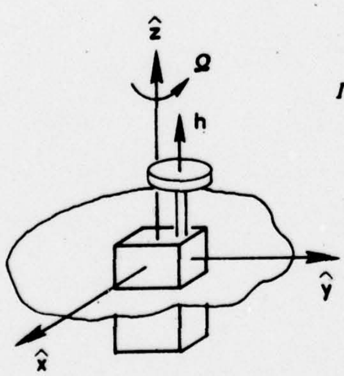
$k_y = \infty$

$$\left(\frac{\omega_x}{\Omega}\right)^2 > \frac{m\,\bar{\Gamma}_z^2}{(C-A)} - \frac{4m\,\bar{\Gamma}_x^2}{C}$$

$$\left(\frac{\omega_z}{\Omega}\right)^2 > \frac{m\,\bar{\Gamma}_y^2\,(C-A) + \bar{\Gamma}_x^2\,(C-B)}{D} - \frac{\left[\dfrac{C\,m^2\,\bar{\Gamma}_y^2\,\bar{\Gamma}_z^2}{C\left(\frac{\omega_x}{\Omega}\right)^2 + 4m\,\bar{\Gamma}_x^2}\right]}{D}$$

where

$$D = (C-B)\left[(C-A) - \frac{Cm\,\bar{\Gamma}_z^2}{C\left(\frac{\omega_x}{\Omega}\right)^2 + 4m\,\bar{\Gamma}_x^2}\right]$$



$\Gamma_{\hat{z}} = 0$

First Mode

$C + \frac{h}{\Omega} > A$ and $C + \frac{h}{\Omega} > B$

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\delta_{x1}^2}{(C + \frac{h}{\Omega} - A)} + \frac{\delta_{y1}^2}{(C + \frac{h}{\Omega} - B)}$$

SYMBOLS

Operator

$\frac{{}^{N}d^2}{dt^2}(\ ) = (\ddot{\ })$	second inertial time derivative of quantity in parentheses
$\frac{{}^{P}d}{dt}(\ ) = (\overset{\circ}{\ })$	time derivative of quantity in parentheses with respect to a rotating frame of reference P
$(\tilde{\ })$	skew symmetric operator that transforms a (3×1) matrix into a (3×3) matrix; Eq. (16)
$(\ )^T$	transpose operation on a matrix

Scalars

A, B, C	principal inertias of system in equilibrium with respect to point 0 referenced to the $\hat{x}$, $\hat{y}$, and $\hat{z}$ axes respectively
A′, B′, C′	principal inertias of rigid portion with respect to point 0 referenced to the $\hat{x}$, $\hat{y}$, and $\hat{z}$ axes respectively
$(k/m)^{1/2}$	unloaded frequency of oscillation reflecting elastic restoring force effects only
m_i	mass of particle i
m_T	total mass of spacecraft
p	mass per unit length of uniform boom
Γ_y	distance of particle in its equilibrium orientation from point 0 ($\underline{r}^1 = \Gamma_y \hat{y}$)
σ	loaded frequency for radially mounted subsystem $(k/m + \Omega^2)^{1/2}$
$\hat{\sigma}$	lowest unloaded natural frequency of uniform boom

SYMBOLS (Continued)

Vectors

$\underline{F}_e^i$	portion of contact forces applied to particle i due to effects of elasticity
$\sum_j \underline{F}_j^i$	vector sum of all contact forces applied to particle i (j is an arbitrary summation index)
$\underline{F}'^i$	portion of contact forces applied to particle i due to effects of spin; also known as preload effect
$\underline{H}^c$	total angular momentum of composite system referenced to composite center of mass
$\underline{r}_i$	nominal equilibrium position of particle i with respect to point 0
$\underline{x}_i$	instantaneous position of particle i with respect to Newtonian frame of reference
$\underline{\delta}_i$	small deformation of particle i from its equilibrium position
$\underline{\eta}$	vector sum of particle deformations from equilibrium
$\underline{\rho}_o$	vector emanating from ccm and terminating at point 0
$\underline{\xi}$	vector difference of particle deformations from equilibrium
$\underline{\omega}$	inertial angular velocity of P frame of reference
$\underline{\delta\omega}$	variational angular velocity from pure spin; Eq. (9)
$\underline{\Omega}$	nominal inertial angular velocity of $\{\hat{x}, \hat{y}, \hat{z}\}$ when system is in equilibrium orientation; assumed to be a state of constant spin ($\underline{\Omega} = \Omega\hat{z}$)

Dyadics

$\mathbf{E}$	unit dyadic ($\mathbf{E} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$)

SYMBOLS (Continued)

${}^{o}\mathbf{I}$	inertia dyadic of hub referenced to point 0
${}^{o}\mathbf{J}$	total system inertia dyadic referenced to point 0
${}^{o}\overset{o}{\mathbf{J}}$	time derivative of total system inertia dyadic with respect to a platform fixed basis P

<u>Matrices</u>

q	matrix (N × 1) of z axis displacements
{x}	(3 × 1) matrix, the elements of which are unit vectors $[\hat{x}, \hat{y}, \hat{z}]^T$
Φ	matrix (N × N) of eigenvectors obtained from eigenvalue problem associated with homogeneous matrix equation $(M\ddot{q} + Kq = 0)$

<u>Other</u>

ccm	composite system center of mass
0	system center of mass when system is in equilibrium orientation
$\{\hat{x}, \hat{y}, \hat{z}\}$	orthonormal vector basis set fixed in rigid portion of spacecraft; referred to as P frame of reference